Finding a Best Fit Plane to Non-coplanar Point-cloud Data Using Non Linear and Linear Equations

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Abstract

In recent times, dimensional model fitting has gained wide acceptance in various fields of manufacturing including co-ordinate metrology, reverse engineering and computer / machine vision. In this paper, a methodology of least square technique is discussed which can be used to find the equation of a best fit plane for non-coplanar points obtained by laser scanning of an object. The theoretical method to get the equation of best fit plane includes a solution of non-linear homogeneous set of equations which are not easy to tackle and as such some method is to be developed to convert the non-linear set of homogeneous set of equations into linear simultaneous equations. A case study is also introduced to explain the procedure to handle linear as well as non-linear simultaneous equations. These set of linear simultaneous equations can then be used in various applications like fast surface generation in reverse engineered projects and even in the field of computer aided inspection for surface roughness measurement using light sectioning vision system where similar approach can be used.

Keywords: *Reverse Engineering, Least square technique, Linear homogeneous simultaneous equation, Computer aided inspection.*

1 Introduction

The concept of best fit curve to an experimental data is a well known concept. However in reverse engineering point cloud data is available as basic input, and a surface generation from this point cloud is the final output [1,2]. Alternatively a method for probabilistic plane fitting in an orthogonal least square sense is used for 3D mapping applications [3]. While generating surface model a plane can be used to fit a surface on a selected group of points. This paper presents a method to find the equation of best fit plane to a group of points obtained by laser scanning of an object using Least Square Technique.

The concept of best fit curve suggests that if an experimental data is available described by an independent variable, and a dependent variable, the points are plotted on x-y plane and a smooth curve is drawn by judgment in such a way that all points are approximately distributed around the curve equally. Mathematically the distance between the actual point and the corresponding point on the curve drawn is known as deviation. Suppose x_1, x_2, x_3, ..., x_n are the values of independent variable, and let y_1, y_2, y_3, ..., y_n be the values of corresponding dependent variable, then,

$$\text{Let } \bar{y} = f(x) \tag{1}$$

be the approximation to the function.

Let 'δ' denote the deviation that is the difference between y and $\bar{y}$ such that:

$$\begin{aligned} \delta_1 &= y_1 - \bar{y}_1 = y_1 - f(x_1) \\ \delta_2 &= y_2 - \bar{y}_2 = y_2 - f(x_2) \\ &\vdots \\ \delta_n &= y_n - \bar{y}_n = y_n - f(x_n) \end{aligned} \tag{2}$$

The function f(x) is to be chosen such that the sum of square of deviations $\delta_1, \delta_2, \delta_3 \ldots\ldots..\delta_n$ should be as small as possible.

Following criteria is used for minimization of deviation.

$$min \sum (\delta_i)^2 = min \sum (\bar{y}_i - y_i)^2 \quad (3)$$

The same concept may be expanded to fit a plane to an available point cloud, obtained on reverse engineered product. One of the prerequisite to randomly distributed point cloud is the process of segmentation before the extraction of spatial information [4]. To ensure that the least square technique gives the optimum results, it is to be ascertained that outliers are eliminated using segmentation techniques. Though least square fitting for estimating the normals at all sample points of a point- cloud data set in the presence of noise is studied, analysis allows for finding neighbourhood size [5].

2 Procedure-I

The general equation of a plane is given by:

$$ax + by + cz + d = 0$$

where x, y, z are the values of x, y and z coordinate of a point on plane and a, b, c, d are four constants, where d denotes the length of normal from origin.

The equation of best fit plane can be formulated as follows:

Let the equation of plane be

$$Ax + By + Cz + D = 0 \quad (4)$$

where A, B, C, D are four unknown constants to be determined.

Let $y_{(act)}$ and $z_{(act)}$ be the actual coordinates of a point on the plane. The theoretical value of x coordinate can be calculated by:

$$x_{th} = \frac{-By_{(act)} - Cz_{(act)} - D}{A} \quad (5)$$

The error or deviation can be evaluated by:

$$err = x_{(act)} - x_{(th)}$$

Therefore,

$$err = x_{(act)} + \frac{By_{(act)} + Cz_{(act)} + D}{A} \quad (6)$$

Sum of error for 'n' points can be evaluated as:

$$\sum_{i=1}^{n} err \sum_{i=1}^{n} \left[x_{(act)} + \frac{By_{(act)} + Cz_{(act)} + D}{A} \right] \quad (7)$$

As errors can be positive or negative the sum of squares of errors is considered. The sum of squares of errors can be evaluated by:

$$(err_i)^2 = \sum_{i=1}^{n} \left(\frac{Ax_i + By_i + Cz_i + D}{A}\right)^2 = 0 \quad (8)$$

To minimize square of errors the partial derivative of square of errors with respect to A, B, C, D should be equated to zero. Hence,

$$\frac{\partial (err_i)^2}{\partial A} = \sum_{i=1}^{n} \left[2\left(\frac{Ax_i + By_i + Cz_i + D}{A}\right) \times \frac{1}{A^2} \times (By_i + Cz_i + D) \right] = 0 \quad (9)$$

After solving equation (9), following equation (10) is obtained.

$$\frac{\partial (err_i)^2}{\partial A} = \frac{2}{A^3} \sum_{i=1}^{n} [(ABx_i y_i + ACx_i z_i + ADx_i + (By_i)^2 + BCy_i z_i + BDy_i + BCy_i z_i + (Cz_i)^2 + CDz_i + BDy_i + CDz_i + D^2)] = 0$$

Resolving further,

$$\frac{\partial (err_i)^2}{\partial A} = \frac{2}{A^3} [AB \sum_{i=1}^{n} x_i y_i + AC \sum_{i=1}^{n} x_i z_i + AD \sum_{i=1}^{n} x_i + B^2 \sum_{i=1}^{n} (y_i)^2 + BC \sum_{i=1}^{n} y_i z_i + BD \sum_{i=1}^{n} y_i + BC \sum_{i=1}^{n} y_i z_i + C^2 \sum_{i=1}^{n} (z_i)^2 + CD \sum_{i=1}^{n} z_i + BD \sum_{i=1}^{n} y_i + CD \sum_{i=1}^{n} z_i + D^2] = 0 \quad (10)$$

Similarly, it can be proved for constant B,

$$\frac{\partial (err_i)^2}{\partial B} = \frac{2}{A^3} [AB \sum_{i=1}^{n} x_i + AC \sum_{i=1}^{n} x_i z_i + B^2 \sum_{i=1}^{n} y_i + BC \sum_{i=1}^{n} y_i z_i + BC \sum_{i=1}^{n} z_i + C^2 \sum_{i=1}^{n} (z_i)^2 + BD + CD \sum_{i=1}^{n} z_i] = 0 \quad (11)$$

And with respect to constant C

$$\frac{\partial(err_i)^2}{\partial C} = \frac{2}{A^3}[AB\sum_{i=1}^{n} x_i y_i + AC\sum_{i=1}^{n} x_i + B^2\sum_{i=1}^{n}(y_i)^2 + BC\sum_{i=1}^{n} y_i + BC\sum_{i=1}^{n} y_i z_i + C^2\sum_{i=1}^{n} z_i + BD\sum_{i=1}^{n} y_i + CD] = 0 \quad (12)$$

Also also with respect to constant D

$$\frac{\partial(err_i)^2}{\partial D} = \frac{2}{A^3}[AB\sum_{i=1}^{n} x_i y_i + AC\sum_{i=1}^{n} x_i z_i + A\sum_{i=1}^{n} x_i + B^2\sum_{i=1}^{n}(y_i)^2 + BC\sum_{i=1}^{n} y_i z_i + B\sum_{i=1}^{n} y_i + BC\sum_{n=1}^{n} y_i z_i + C^2\sum_{n=1}^{n} {z_i}^2 + C\sum_{n=1}^{n} z_i + BD\sum_{n=1}^{n} y_i + CD\sum_{n=1}^{n} z_i + D] = 0 \quad (13)$$

This gives a set of four non-linear homogeneous equations in 10, 11, 12 and 13 containing four unknown constants A, B, C and D.

These equationscan be solved using Newton's iterative method and the values of four unknown constants A, B, C and D can be obtained.

2.1 *Case Study I*

An example is assumed to verify the above set of equations. Table 1 gives the information about the co-ordinates of seven points. To validate the above derived equations initially an equation for one plane is assumed. Table gives the x, y, z co-ordinates of seven co-planar points. It also gives $\sum x$, $\sum y$, $\sum z$, $\sum xy$, $\sum xz$, $\sum yz$, $\sum x^2$, $\sum y^2$ and $\sum z^2$.

Consider the equation of plane is:

x + 2y + 4z = 8. Hence A = 1, B = 2, C = 4 and D = -8 as A, B, C and D indicates four unknown constants from equation of a plane.

Table 1: Case Study I

	X	y	z	xy	xz	yz	x^2	y^2	z^2
1	8	0	0	0	0	0	64	0	0
2	4	0	1	0	4	0	16	0	1
3	6	1	0	6	0	0	36	1	0
4	0	0	2	0	0	0	0	0	4
5	-2	1	2	-2	-4	2	4	1	4
6	4	2	0	8	0	0	16	4	0
7	0	2	1	0	0	2	0	4	1
∑	**20**	**6**	**6**	**12**	**0**	**4**	**136**	**10**	**10**

Using equation 10, following equation is formulated.

$$\frac{\partial(err_i)^2}{\partial A} = \begin{pmatrix} 12AB + 0AC + 20AD + 10BB + 4BC + 6BD \\ + 4BC + 10CC + 6CD + 6BD + 6CD + 7D^2 \end{pmatrix}$$

Putting A = 1, B = 2, C= 4 and D = -8 in the above equation, the result obtained is:

$$\frac{\partial(err_i)^2}{\partial A} = \begin{pmatrix} 24 - 160 + 40 + 64 - \\ 192 + 160 - 384 + 448 \end{pmatrix} = 0$$

which satisfies equation 11 as, for constant A, in equation 11 value of $\frac{\partial(err_i)^2}{\partial A}$ is obtained equal to zero.

Similarly using equation 11, 12 and 13 three more equations are formulated for $\frac{\partial(err_i)^2}{\partial B}$, $\frac{\partial(err_i)^2}{\partial C}$ and $\frac{\partial(err_i)^2}{\partial D}$ and solved putting A = 1, B = 2, C= 4 and D = -8.

$$\frac{\partial(err_i)^2}{\partial B} = \begin{pmatrix} 20AB + 0AC + 6B^2 + 4BC + 6BC + \\ 10C^2 + 7BD + 6CD \end{pmatrix}$$

$$\frac{\partial(err_i)^2}{\partial B} = \begin{pmatrix} 40 + 24 + 80 + 160 - 112 - 192 \\ \end{pmatrix} = 0$$

which satisfies equation 11.

$$\frac{\partial (err_i)^2}{\partial C} = \begin{pmatrix} 12AB + 20AC + 10B^2 + 10BC \\ + 6C^2 + 6BD + 7CD \end{pmatrix}$$

$$\frac{\partial (err_i)^2}{\partial C} = \begin{pmatrix} 24 + 80 + 40 + 80 + 96 \\ - 96 - 224 \\ \end{pmatrix} = 0$$

which satisfies equation 12.

$$\frac{\partial (err_i)^2}{\partial D} = \begin{pmatrix} 12AB + 20A + 10B^2 + 8BC + 6B + \\ 10C^2 + 6C + 6BD + 6CD + 7D \end{pmatrix}$$

$$\frac{\partial (err_i)^2}{\partial B} = \begin{pmatrix} 24 + 20 + 40 + 64 + 12 + \\ 160 + 24 - 96 - 192 - 56 \\ \end{pmatrix} = 0$$

which satisfies equation 13.

3 Procedure-II

Above case study presents a theoretical method to get the equation of best fit plane. However it includes a solution of non-linear homogeneous set of equations which are not easy to tackle and as such some method is to be developed to convert the non linear set of homogeneous set of equations into linear simultaneous equations to find the best fit plane. This method is given below with suitable illustration.

Consider the equation of error as represented in equation (6), that is:

$$err = X_{(act)} + \frac{By_{(act)} + Cz_{(act)} + D}{A}$$

Sum of error for ‘n’ points can be evaluated as using equation (8) and (9)

$$\sum_{1}^{n} err = \sum_{1}^{n} \left[X_{(act)} + \frac{By_{(act)} + Cz_{(act)} + D}{A} \right]$$

As errors can be positive or negative the sum of squares of errors is considered. The sum of squares of errors can be evaluated by,

$$(err_i)^2 = \sum_{i=1}^{n} \left(\frac{Ax_i + By_i + Cz_i + D}{A} \right)^2$$

Let B/A = I, C/A = J, and D/A = K therefore,

$$\sum_{i=1}^{n} = \sum_{i=1}^{n} (x_i + Iy_i + Jz_i + K)^2 = 0 \tag{8a}$$

To minimize error the partial derivative of $(err)^2$ with respect to I, J, K should be equated to zero.

Differentiating equation (8a) with respect to I, J, K and equating to zero,

$$\frac{\partial (err_i)^2}{\partial I} = 2 \times \left(\sum_{i=1}^{n} x_i y_i + I \sum_{i=1}^{n} (y_i)^2 + J \sum_{i=1}^{n} y_i z_i + K \sum_{i=1}^{n} y_i \right) = 0$$

$$\begin{aligned} &= 2 \times (x_1 y_1 + I(y_1)^2 + J y_1 z_1 + K y_1) \\ &+ 2 \times (x_2 y_2 + I(y_2)^2 + J y_2 z_2 + K y_2) \\ &\vdots \\ &+ 2 \times (x_n y_n + I(y_n)^2 + J y_n z_n + K y_n) \end{aligned}$$

Solving above equation, equation obtained is:

$$\sum_{i=1}^{n} x_i y_i = -I \sum_{i=1}^{n} y^2 - J \sum_{i=1}^{n} yz - K \sum_{i=1}^{n} y_i \tag{14}$$

Similarly for $\frac{\partial (err_i)^2}{\partial J}$ and $\frac{\partial (err_i)^2}{\partial k}$ equations obtained are given below:

$$\sum_{i=1}^{n} x_i z_i = -I \sum_{i=1}^{n} y_i z_i - J \sum_{i=1}^{n} {z_i}^2 - K \sum_{i=1}^{n} z_i \tag{15}$$

$$\sum_{i=1}^{n} x_i = -I \sum_{i=1}^{n} y_i - J \sum_{i=1}^{n} z_i - nK \tag{16}$$

3.1 *Case Study-II*

In order to verify the results obtained in Case study I, following case study is presented.

The table 2 shows the x, y, z coordinates of seven points. The data is obtained by assuming the equation of plane to be: x + 2y + 4z = 8.

Various values of y and z are assumed and corresponding values of x are calculated. The actual values of x are evaluated by adding an error of 0.1 to theoretical values of x.

Table 2: Case Study II

[Point Cloud Patch containing 7 points]

	x	y	z	xy	xz	yz	x^2	y^2	z^2
1	8.1	0	0	0	0	0	65.6	0	0
2	3.9	0	1	0	3.9	0	15.2	0	1
3	6.1	1	0	6	0	0	37.2	1	0
4	0.1	0	2	0	0.2	0	0.01	0	4
5	-2.1	1	2	-2.1	-4.2	2	4.41	1	4
6	3.9	2	0	7.8	0	0	15.2	4	0
7	-0.1	2	1	-0.2	-0.1	2	0.01	4	1
∑	**19.9**	**6**	**6**	**11.6**	**0**	**4**	**138**	**10**	**10**

Substituting in equations 14, 15 and 16, one will obtain respectively equations (14a), (15a) and (16a).

$$\sum_{i=1}^{n} x_i y_i = -I \sum_{i=1}^{n} {y_i}^2 - J \sum_{i=1}^{n} y_i z_i - K \sum_{i=1}^{n} y_i \tag{14}$$

$$11.6 = -10I - 4J - 6K \tag{14a}$$

$$\sum_{i=1}^{n} x_i z_i = -I \sum_{i=1}^{n} y_i z_i - J \sum_{i=1}^{n} (z_i)^2 - K \sum_{i=1}^{n} z_i \tag{15}$$

$$-0.2 = -4I - 10J - 6K \tag{15a}$$

$$\sum_{i=1}^{n} x_i = -I \sum_{i=1}^{n} y_i - J \sum_{i=1}^{n} z_i - nK \tag{16}$$

$$19.9 = -6I - 6J - 7K \tag{16a}$$

Solving above equations, values obtained are,

I = 2.0744, J = 4.0410, K = -8.0846

Consider equation of a plane:

Ax + By + Cz + D = 0 and is similar to x + Iy + Jy + k = 0.

Hence, one can write, A = 1, B = 2.0744, C = 4.0410 and D = 8.0846.

Giving rise to equation of best fit plane x + 2.0744y + 4.0410z = 8.0846

which matches with the assumed equation of a plane, that is:

x + 2y + 4z - 8 = 0.

By using this method, an equation of best-fit plane can be obtained for point cloud data that is obtained by LASER scanning wherein the data obtained is scattered and dispersed.

3.2 *Case Study-III*

This is a point cloud data of 16 points which indicates random point cloud patch. Considering these points the equation of best-fit plane is calculated using equations 14, 15 and 16.

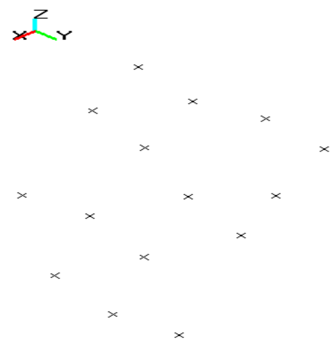

Figure 1: Point Cloud Patch of 16 Points

Solving three equations obained from table 3 which is given below, result obtained are : I = -0.0334, J = -0.6691 and K = 195.8947

Hence the equation of plane is:

$$x + 0.0334y + 0.6691z - 195.8947 = 0 \tag{17}$$

Results obtained by using equation (17)

By using this equation (17), a plane is plotted in Imageware and then this plotted plane is compared with a given point cloud data of 16 points to study the deviation pattern of these 16 points with the so formed plane. The results obtained in Imageware are plotted as shown in figure 2.

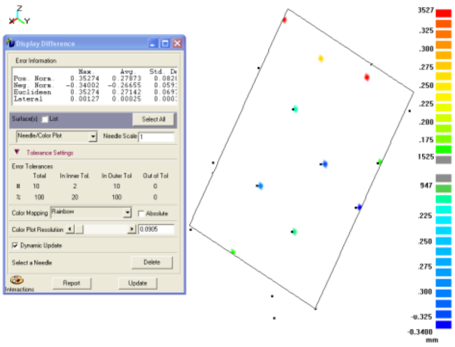

Figure 2: Verification between Plane generated using equation 17 and point cloud patch

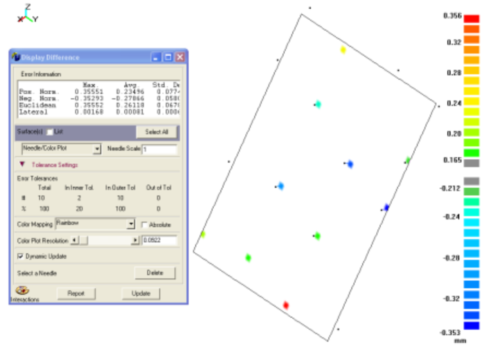

Figure 3: Verification between Plane Generated in Imageware and Point Cloud Patch

Results obtained using equation of a plane fitted by Imageware Software:

By using direct command for best fit plane in Imageware, again another plane is fitted over the same given set of 16 points and the difference between plane and pointclouddata is calculated which is shown in figure 3.

Table 3: Case Study-III

Sl No	x	y	z	x^2	y^2	z^2	xy	xz	yz
1	108.83	209	119.05	11844.69	43681	14173.12	22746.16	12956.70	24881.64
2	110.31	209	117.71	12168.85	43681	13855.90	23055.31	12985.01	24601.62
3	112.56	209	114.63	12670.32	43681	13140.43	23525.56	12903.23	23958.03
4	113.68	209	112.13	12922.01	43681	12572.85	23758.08	12746.23	23434.90
5	109.08	211	118.75	11899.17	44521	14100.95	23016.58	12953.36	25055.70
6	110.63	211	117.30	12237.89	44521	13759.41	23341.88	12976.37	24750.41
7	112.41	211	114.80	12635.18	44521	13177.94	23717.73	12903.70	24221.79
8	113.53	211	112.30	12887.93	44521	12610.89	23953.78	12748.65	23694.92
9	108.75	213	118.97	11826.56	45369	14154.96	23163.75	12938.49	25341.59
10	111.25	213	116.46	12376.56	45369	13563.21	23696.25	12956.31	24806.24
11	112.67	213	114.14	12693.79	45369	13027.41	23998.01	12859.52	24311.33
12	113.65	213	111.67	12916.32	45369	12470.88	24207.45	12691.65	23786.37
13	108.88	215	118.81	11853.77	46225	14115.25	23408.13	12935.18	25543.63
14	110.44	215	117.34	12196.44	46225	13768.09	23744.06	12958.46	25227.56
15	111.54	215	115.95	12441.55	46225	13444.10	23981.47	12933.12	24928.97
16	113.53	215	111.88	12887.93	46225	12517.40	24407.88	12701.31	24054.46
$\sum$	1781.72	3392	1851.89	198458.96	719184	214452.79	377722.08	206147.29	392599.16
377722.1 = – (I*719184) – (J*392599.1) – (K*3392)									
206147.3 = – (I*392599.1) – (J*214452.8) – (K*1851.885)									
1781.719 = – (I*3392) – (J*1851.885) – (K*16)									

Table 4: Comparaison between two results shown in figure 3 and 4

Sl No		Details of a plane developed using discussed methodology	Details of a plane developed using direct command from Imageware	Remark
1	Equation of a plane	X + (0.0334)Y + (0.6691)Z - 195.8947 = 0	0.8X + 0.01Y + 0.6Z - 0.2 = 0	As seen from the deviation the eq^{n}. (17) so developed is validated as the deviation results obtained using eq^{n}. (17) tally with results obtained using equation developed using direct command for best fit plane from Imageware.
2	Average deviation obtained between plane and given set of points	Positive Norm: 0.27873 Negative Norm: -0.26655	Positive Norm: 0.23496 Negative Norm: -0.27866	

4 Conclusions

By using this proposed method an equation of best-fit plane can be constructed for sampled point cloud data that has been obtained from a reverse engineered product.

Once the point data is segmented, appropriate surfaces can be fitted using best fit plane method over the almost coplanar segmented point cloud data and the entire surface model is constructed by extending these surfaces and finding the intersection of curves between them.

Using best fit plane approach, segmentation of point cloud process becomes simple. In this technique point cloud set is segmentated into noumber of point cloud patches and every patch contains coplanr points. By keeping boundary points of individual patch as it is one can achieve data reduction by elliminating points lying inside the boundary.

Also the best fit plane can be extended to a set of point cloud data which can help to formulate fast generation of surfaces for mapping 3D applications and also creation of STL files for reverse engineered products.

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