Research Article

Design and Application of a Modified EWMA Control Chart for Monitoring Process Mean

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Abstract
The modified exponentially weighted moving average (modified EWMA) control chart is an improvement on the performance of the standard EWMA control chart for detecting small and abrupt shifts in the process mean. In this study, the effect of varying the constant and exponential smoothing parameters for detecting shifts in the mean of an autoregressive process with exogenous variables (ARX(p,r)) with a trend and exponentially distributed white noise on the standard and modified EWMA control chart was investigated. The performances of the two control charts were compared via their average run lengths (ARLs) computed by using explicit formulas and the numerical integrated equation (NIE) technique. A comparative study of the two ARL methods on the modified and traditional EWMA control charts shows that the modified schemes had the better detection ability at all levels of shift sizes. Finally, two examples using real datasets on gold and silver prices are given to illustrate the applicability of the proposed procedure. Our findings advocate that the modified EWMA chart is excellent for monitoring ARX(p,r) processes with exponentially distributed white noise

Keywords: Autoregressive, Autocorrelation, Exogenous variable, Explicit formula, SPC, ARL

1 Introduction

Gold is a precious metal that is usually yellow, visually pleasing, workable, and does not tarnish or corrode. Moreover, 75% of all gold is used in the production of jewelry, an estimated 60% of which is in China and India, and it is used as a traditional gift in many cultures and often given during festivals such as the Chinese New Year, Diwali, and Ramadan Eid [1]. Furthermore, gold has practical uses, such as in electronics, transportation, communications, and medical technology. Economic wealth and foreign exchange reserves are often in the form of gold as it is perceived as a commodity with low economic risk and immune to fluctuations in inflation, currency exchange, and economic and political crises.

Many factors affect the price of gold, such as foreign exchange rates against the Euro and the US dollar, the US gross domestic product (GDP) [2], as well as the prices of crude oil, platinum, silver, and palladium. However, during global financial crises, many investors buy riskier alternatives to gold. Because gold is rooted in behavioral bias associated with its historical and current high value, its perceived role as a safe haven [3], so it is expected to retain or increase its value during times of market turbulence. When the markets downturn, investors try to limit their exposure to losses by investing in gold.

Various factors affecting movements in financial markets make an influence of the price of gold all the time. As one of the most profusely purchased investment products for many investors, the gold price is growingly attracting attention. Hence, it should be the priority to follow this situation under control and closely monitored periodically.

Statistical process control is a widely used approach used to monitor processes by detecting changes in the process mean. Economic and financial data are often autocorrelated [4]–[9], so the classical assumption for control charts that the process observations follow a normal distribution with random white noise is violated. Control charts for monitoring autocorrelated observation are the cumulative sum (CUSUM) [10] and exponentially weighted moving average (EWMA) [11] control charts, which are good for detecting small changes in the process mean for autocorrelated observations. Later, Patel and Divecha [12] changed the structure of the EWMA statistic to produce the modified EWMA control chart, which was then generalized by Khan *et al.* [13]. It has been claimed that the modified EWMA control chart performs well for changes in the process mean for observations following non-normal distributions [14]–[16], which has also been illustrated with the real-life examples by comparison with the existing EWMA control chart. It has been reported in several studies that the modified EWMA procedure is excellent for small and abrupt changes in the process mean, as investigated by using the average run length (ARL) [17]–[20].

Explicit formulas can be used to derive the exact value of the ARL, and many researchers have used them to evaluate the ARL when the observations are autocorrelated with exponentially distributed white noise. Paichit [21] proposed an analytical expression for the ARL on a CUSUM control chart when observations are from a first-order autoregressive (AR) process with explanatory variables (ARX(1)) with exponentially distributed white noise. Areepong [22] studied the performance of the EWMA control chart for monitoring an ARX process by using the ARL based on explicit formulas. Later, the explicit formulas of ARL for an autoregressive moving average process with an exogenous variable (ARMAX) process with exponentially distributed white noise on a CUSUM control chart was studied by Peerajit and Areepong [23]. Sunthornwat and Areepong [24] derived the ARL on the CUSUM control chart for seasonal and non-seasonal MA processes with exogenous variables using explicit formulas and validated it against the NIE method. Recently, the detection capability of the modified EWMA control chart for a stationary AR(1) model with a trend and exponentially distributed white noise using the ARL based on explicit formulas was evaluated by comparing it with the numerical integral equation (NIE) technique [25].

In this paper, the derivation of the explicit formulas of the ARL on the modified EWMA control chart for an ARX(p,r) process with a trend and exponentially distributed white noise are proposed and the numerical results using the ARL on this basis are reported. Moreover, we also compared the performances of the ARL on the modified and standard EWMA control charts. The applicability of the modified EWMA chart using the ARL based on explicit formulas was demonstrated using real data on gold and silver prices.

2 Design of the Modified EWMA Control Chart

According to the Shewhart type control chart, all information available in the current sample is used to establishing the control statistic. Whereas the EWMA control procedure is considered in the term the previously statistic that is given the most weight and the current of the observations are given geometrically decreasing weights.

Let $Y_1, Y_2, ..., Y_t$ be a sequence of observation with mean μ and variance σ^2. The EWMA statistic can be written with the smoothing constant λ as

$$M_t = (1-\lambda)M_{t-1} + \lambda Y_t,\ 0 < \lambda \le 1. \quad (1)$$

If λ = 1, the EWMA statistic reduces to the Shewhart statistic. For in control process the mean and variance of EWMA statistic are given by

$$E(M_t) = \mu,$$

$$V(M_t) = \sigma^2 \frac{\lambda}{2-\lambda}.$$

The range value of parameter $0.05 \le \lambda \le 0.25$ is recommended for well working in practice [26].

Therefore, the lower control limit (LCL) and upper control limit to detect the observation with a suitable control width limit K are as follow

$$LCL = \mu - K\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \quad (2)$$

$$UCL = \mu + K\sigma\sqrt{\frac{\lambda}{2-\lambda}}. \quad (3)$$

Subsequently, the extended EWMA statistic was proposed by Patel and Divecha [12] namely modified EWMA control chart. It was also structured the generalized of control statistic by Khan *et al.* [13]. The new modified EWMA control chart is efficient in the order of detecting the observations that are independently normal distributed or autocorrelation. The modified EWMA control is based on the statistic with the smoothing constant λ

$$M_t = (1-\lambda)M_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1}), \quad (4)$$

where $0 < \lambda \le 1$ and k is a constant, which should be given a small value.

The in control process of mean and variance of modified EWMA control chart are

$$E(M_t) = \mu,$$

$$V(M_t) = \sigma^2 \frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}.$$

Hence, the LCL and UCL with an appropriate control width limit L are given as follows

$$LCL = \mu - L\sigma\sqrt{\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}}, \quad (5)$$

$$UCL = \mu + L\sigma\sqrt{\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}}. \quad (6)$$

3 Derivation of the Explicit Formula of ARL

The ARL is a usual measure used to evaluate the efficiency of a control chart. The observations were considered in such a way that is an autoregressive time trend process with exogenous variable denoted by ARX(p,r) model. It can be expressed as follows

$$Y_t = \alpha + \theta T_t + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \ldots + \alpha_p Y_{t-p} + \sum_{i=1}^{r} b_i X_i + \varepsilon_t, \quad (7)$$

where a and $a_1, a_2, \ldots, a_p$ represent the mean and the order of coefficients of the process, respectively θ is a slope, X_i ; $i = 1,2,\ldots,r$ is exogenous or independent of r variables with coefficient b_i ; $i = 1,2,\ldots,r$ that are effecting the observation $Y_1, Y_2, \ldots, Y_t$ and ε_i ; $i = 1,2,\ldots$ represent the white noise which is assumed to be exponential distribution.

The modified EWMA control chart following ARX(p,r) process by Equation (7), the result can be present as

$$M_t = (1-\lambda)M_{t-1} + (\lambda+k)(\alpha + \theta T_t) + (\lambda+k)\alpha_1 Y_{t-1} + (\lambda+k)\alpha_2 Y_{t-2} \ldots + (\lambda+k)\alpha_p Y_{t-p} + (\lambda+k)\sum_{i=1}^{r} b_i X_i + (\lambda+k)\varepsilon_t - kY_{t-1}.$$

For $t = 1$, we have

$$M_1 = (1-\lambda)M_0 + (\lambda+k)(\alpha+\theta T_t) + (\lambda+k)\alpha_1 Y_0 + (\lambda+k)\alpha_2 Y_{1-2} + \ldots + (\lambda+k)\alpha_p Y_{t-p} + (\lambda+k)\sum_{i=1}^{r} b_i X_i + (\lambda+k)\varepsilon_1 - kY_0.$$

If ε_1 give an in-control state for M_1 of one-sided, then $0 \le M_1 \le$ b ; 0 and b are LCL and UCL, respectively. The inequality of ε_1 can be written as, respectively

$$\frac{-A}{\lambda+k} \le \varepsilon_1 \le \frac{b-A}{\lambda+k},$$

where

$$A = (1-\lambda)M_0 + (\lambda+k)(\alpha+\theta T_t) + (\lambda+k)\alpha_1 Y_0 + \ldots + (\lambda+k)\alpha_p Y_{t-p} + (\lambda+k)\sum_{j=1}^{r} b_j X_j - kY_0.$$

Let $R(u)$ be the ARL of the integral equation of the modified EWMA control chart for the ARX(p,r) process. By the method of Champ and Rigdon [27], the integral equation can be written in the form as

$$R(u) = 1 + \int_{\frac{-A}{\lambda+k}}^{\frac{b-A}{\lambda+k}} R\left[\begin{array}{l}(1-\lambda)M_0 + (\lambda+k)(\alpha+\theta T_t) + \\ (\lambda+k)\alpha_1 Y_0 + \ldots + (\lambda+k)\alpha_p Y_{t-p} + \\ (\lambda+k)\sum_{j=1}^{r} b_j X_j - kY_0\end{array}\right] f(g)dg.$$

By changing the variable, we obtain the integral equation as follows:

$$R(u)=1+\frac{1}{\lambda+k}\int_0^b R(g)f\begin{bmatrix} g-(1-\lambda)M_0- \\ (\lambda+k)(\alpha+\theta T_t)- \\ (\lambda+k)\alpha_1 Y_0-\ldots \\ -(\lambda+k)\alpha_p Y_{t-p}+ \\ (\lambda+k)\sum_{j=1}^{r} b_j X_j-kY_0 \end{bmatrix}/(\lambda+k)\,dg.$$

In this study, we assume ε_i is an exponential distribution with parameter β. Therefore, the $R(u)$ is a Fredholm integral equation of the second kind as following

$$R(u)=1+\frac{1}{\lambda+k}\begin{bmatrix}\int_0^b R(g)\frac{1}{\beta}e^{\frac{-g+(1-\lambda)M_0+(\lambda+k)\alpha_1Y_0+\ldots+(\lambda+k)\alpha_pY_{t-p}}{\beta(\lambda+k)}}\cdot \\ e^{\frac{(\alpha+\theta T_t)+\sum_{j=1}^{r}b_jX_j-kY_0}{\beta}}\end{bmatrix}dg.$$

Suppose that

$$P(u)=1+\frac{e^{\frac{(1-\lambda)M_0}{\beta(\lambda+k)}+\frac{(\lambda+k)\alpha_1Y_0+\ldots+(\lambda+k)\alpha_pY_{t-p}\alpha+\theta+\sum_{j=1}^{r}b_jX_j-kY_0}{\beta}}}{\beta(\lambda+k)},$$

and

$$Q=\int_a^b R(g)e^{\frac{-g}{\beta(\lambda+k)}}dg.$$

The integral equation $R(u)$ is rewritten in the form as

$$R(u)=1+\frac{P(u)}{\beta(\lambda+k)}Q.$$

Deriving the solution of Q, we then obtain the one-sided explicit formula of ARL on modified EWMA control chart for the ARX(p,r) process as follows

$$ARL=\frac{\lambda e^{\frac{(1-\lambda)M_0}{\beta(\lambda+k)}}\left[e^{\frac{-b}{\beta(\lambda+k)}}-1\right]}{\lambda e^{\frac{kY_0}{\beta(\lambda+k)}-\frac{(\alpha+\theta T)}{\beta}-\frac{\sum_{i=1}^{p}\alpha_pY_{t-i}}{\beta}-\frac{\sum_{j=1}^{r}b_jX_j}{\beta}}+e^{\frac{-\lambda b}{\beta(\lambda+k)}}-1}, \quad (8)$$

with in-control process parameter β_0 and out-of-control process parameter $\beta_1>\beta_0$.

In addition, the results of explicit formula for evaluation the ARL will be accuracy by the numerical integral equation (NIE) method by the integral equation as follows:

$$R(u)=1+\frac{1}{\lambda+k}\int_0^b R(g)f\left[\frac{g-(1-\lambda)M_0-kY_0}{\lambda+k}\right]-\begin{bmatrix}(\alpha+\theta T_t) \\ +\alpha_1Y_0+\ldots \\ +\alpha_pY_{t-p} \\ +\sum_{j=1}^{r}b_jX_j\end{bmatrix}dg.$$

It is approximated by using the Gauss- Legendre's rule. It can be expressed by

$$\int_0^b R(g)f(g)dg\approx\sum_{j=1}^{m}w_jf(a_j).$$

The integral equation can be approximated by

$$R(a_i)=1\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{R}(a_j)f\begin{Bmatrix}\frac{a_j-(1-\lambda)a_i+kY_0}{(\lambda+k)}-(\alpha+\theta T) \\ -\sum_{i=1}^{p}\alpha_pY_{t-i}-\sum_{j=1}^{r}b_jX_j\end{Bmatrix}$$

$; i=1,2,3,\ldots,m.$

Let $1_{m\times1}=(1,1,\ldots,1)'$ is a column vector of one and let $\mathbf{R}_{m\times m}$ is a matrix and define the $(m,m)^{th}$ is an element of matrix $\mathbf{R}$ as follows

$$[R_{ij}]=\frac{1}{\lambda+k}w_jf\begin{Bmatrix}\frac{a_j-(1-\lambda)u+kY_0}{(\lambda+k)}-(\alpha+\theta T)-\sum_{i=1}^{p}\alpha_pY_{t-i} \\ -\sum_{j=1}^{r}b_jX_j\end{Bmatrix}.$$

Let $\mathbf{I}_{m\times1}=diag(1,1,\ldots,1)$ is a unit matrix order m. If $(\mathbf{I}-\mathbf{R})^{-1}$ there exists, the numerical approximation for the integral equation in a term of matrix as follows $\mathbf{L}_{m\times1}=(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times1}$.

Finally, we substitute a_i by u in $\tilde{R}(a_i)$, the numerical integration equation for function $R(u)$ as follows:

$$\tilde{R}(u)=1\frac{1}{\lambda+k}\sum_{j=1}^{m}w_j\tilde{R}(a_j)f\begin{Bmatrix}\frac{a_j-(1-\lambda)u+kY_0}{(\lambda+k)}-(\alpha+\theta T) \\ -\sum_{i=1}^{p}\alpha_pY_{t-i}-\sum_{j=1}^{r}b_jX_j\end{Bmatrix}.$$

where $a_j=\frac{b}{m}\left(j-\frac{1}{2}\right)$ and $w_j=\frac{b}{m}; j=1,2,\ldots,m.$

In this paper, the initial value Y_0 is set to be v. The numerical results of this study is presented in the next section.

4 Experimental Results

In this section, the evaluation of ARL for ARX(p,r) process with exponential white noise on the modified EWMA control chart is presented. The numerical results of theoretical evaluation of ARL is reported in Section 4.1, and the real application of comparative performance on the modified EWMA chart against the original EWMA control chart is provided in Section 4.2.

4.1 *Numerical results*

The evaluation of ARLs using the explicit formulas and numerical integral equation method, which measure the accuracy in the comparative study are reported in Table 1. The experimental results are computed by using MATHEMATICA program [28]. It is executed by varying parameters of modified EWMA control chart (λ = 0.1, 0.2 and k = 0.5, 1, 1.5, 2) and the trend

Table 1: Comparison of ARL value for trend ARX(p,r) with exogenous variable process on the modified EWMA chart by using explicit formulas and NIE method

Models	Parameters	λ	δ	k = 0.5		k = 1		k = 1.5		k = 2	
				Explicit	NIE	Explicit	NIE	Explicit	NIE	Explicit	NIE
Trend ARX (1,1)	α = 3, α_1 = 0.3, θ = 0.5,	0.1*	0.00	370.0689	370.0685	370.2105	370.2099	370.0107	370.0099	370.1918	370.1908
			0.01	70.4030	70.4030	40.7531	40.7531	32.8403	32.8403	29.2873	29.2873
			0.03	26.0252	26.0252	14.6596	14.6596	11.8254	11.8254	10.5760	10.5759
			0.05	15.6003	15.6003	8.9632	8.9632	7.3209	7.3209	6.5976	6.5976
			0.10	7.5058	7.5058	4.6213	4.6213	3.9008	3.9008	3.5818	3.5818
			0.30	2.3782	2.3782	1.8433	1.8433	1.7006	1.7006	1.6357	1.6357
			0.50	1.5759	1.5759	1.3795	1.3795	1.3242	1.3242	1.2985	1.2985
		0.2**	0.00	370.0737	370.0727	370.2947	370.2928	370.1113	370.1088	370.1672	370.1643
			0.01	51.8648	51.8648	35.3352	35.3352	30.0401	30.0401	27.4761	27.4761
			0.03	18.7371	18.7371	12.6961	12.6961	10.8310	10.8310	9.9386	9.9386
			0.05	11.3112	11.3112	7.8166	7.8166	6.7402	6.7402	6.2252	6.2252
			0.10	5.6204	5.6204	4.1091	4.1091	3.6390	3.6390	3.4129	3.4129
			0.30	2.0136	2.0136	1.7338	1.7338	1.6422	1.6422	1.5971	1.5971
			0.50	1.4351	1.4351	1.3335	1.3335	1.2988	1.2988	1.2814	1.2814
Trend ARX(2,1)	α = 3, α_1 = 0.2, α_2 = 0.4, θ = 0.5,	0.1***	0.00	370.0494	370.0492	370.0131	370.0127	370.1017	370.1012	370.2425	370.2420
			0.01	67.2024	67.2023	38.7177	38.7177	31.1674	31.1674	27.7822	27.7821
			0.03	24.6448	24.6448	13.8722	13.8722	11.1941	11.1941	10.0143	10.0143
			0.05	14.7336	14.7336	8.4761	8.4761	6.9311	6.9311	6.2509	6.2509
			0.10	7.0688	7.0688	4.3736	4.3736	3.7013	3.7013	3.4037	3.4037
			0.30	2.2514	2.2514	1.7651	1.7651	1.6355	1.6355	1.5765	1.5765
			0.50	1.5114	1.5114	1.3368	1.3368	1.2877	1.2877	1.2649	1.2649
		0.2****	0.00	370.0691	370.0685	370.1180	370.1170	370.1720	370.1707	370.2023	370.2007
			0.01	49.4135	49.4135	33.5623	33.5623	28.5074	28.5074	26.0623	26.0623
			0.03	17.7514	17.7514	12.0228	12.0228	10.2581	10.2581	9.4142	9.4142
			0.05	10.6995	10.6995	7.4010	7.4010	6.3867	6.3867	5.9017	5.9017
			0.10	5.3117	5.3117	3.8972	3.8972	3.4578	3.4578	3.2466	3.2466
			0.30	1.9208	1.9208	1.6659	1.6659	1.5825	1.5825	1.5415	1.5415
			0.50	1.3866	1.3866	1.2960	1.2960	1.2651	1.2651	1.2496	1.2496
Trend ARX (2,2)	α = 3, α_1 = 0.1, α_2 = 0.2, θ = 0.5,	0.1*****	0.00	370.0309	370.0309	370.0372	370.0371	370.0248	370.0247	370.1810	370.1809
			0.01	60.6905	60.6905	34.6443	34.6443	27.8278	27.8278	24.7841	24.7841
			0.03	21.8935	21.8935	12.3147	12.3147	9.9472	9.9472	8.9062	8.9062
			0.05	13.0161	13.0161	7.5160	7.5160	6.1635	6.1635	5.5689	5.5689
			0.10	6.2088	6.2088	3.8877	3.8877	3.3103	3.3103	3.0549	3.0549
			0.30	2.0076	2.0076	1.6150	1.6150	1.5105	1.5105	1.4630	1.4630
			0.50	1.3904	1.3904	1.2568	1.2568	1.2192	1.2192	1.2018	1.2018
		0.2******	0.00	370.0763	370.0762	370.0693	370.0691	370.0667	370.0664	370.4406	370.4402
			0.01	44.4673	44.4673	30.0237	30.0237	25.4552	25.4552	23.2537	23.2537
			0.03	15.7925	15.7925	10.6926	10.6926	9.1281	9.1281	8.3813	8.3813
			0.05	9.4886	9.4886	6.5824	6.5824	5.6916	5.6916	5.2662	5.2662
			0.10	4.7043	4.7043	3.4820	3.4820	3.1031	3.1031	2.9212	2.9212
			0.30	1.7423	1.7423	1.5357	1.5357	1.4682	1.4682	1.4350	1.4350
			0.50	1.2954	1.2954	1.2258	1.2258	1.2021	1.2021	1.1902	1.1902

*b=1.12372×10^{-2} for k=0.5, b=2.23563×10^{-2} for k=1, b=3.3502×10^{-2} for k=1.5 and b=4.46577×10^{-2} for k=2.
**b=1.16888×10^{-2} for k=0.5, b=2.26481×10^{-2} for k=1, b=3.37313×10^{-2} for k=1.5 and b=4.48572×10^{-2} for k=2.
***b=8.32248×10^{-3} for k=0.5, b=1.65573×10^{-2} for k=1, b=2.48119×10^{-2} for k=1.5 and b=3.30738×$10^{-2}$for k=2.
****b=8.6554×10^{-3} for k=0.5, b=1.67697×10^{-2} for k=1, b=2.49757×10^{-2} for k=1.5 and b=3.32132×10^{-2} for k=2.
*****b=4.131221×10^{-3} for k=0.5, b=8.21883×10^{-3} for k=1, b=1.23162×10^{-2} for k=1.5 and b=1.64172×10^{-2} for k=2.
******b=4.29538×10^{-3} for k=0.5, b=8.3216×10^{-3} for k=1, b=1.239322×10^{-2} for k=1.5 and b=1.64805×10^{-2} for k=2.

ARX(p,r) process with the in-control process $\beta_0 = 1$; out-of-control process $\beta_1 = \beta_0 + \delta\beta_0$, where $\delta = 0.01$, 0.03, 0.05, 0.10, 0.30 or 0.50 given $ARL_0 = 370$. The parameters of the trend autoregressive process with exogenous variable were set as $a = 3$, $\theta = 0.5$; $a_1 = 0.3$ for trend ARX(1,1) process ; $a_1 = 0.2$, $a_2 = 0.4$ for trend ARX(2,1) model ; $a_1 = 0.1$, $a_2 = 0.2$ for trend ARX(2,2) model with $b_1 = 1$ and $b_1 = 1$ for one and two independent exogenous variables, respectively.

The Table 1 shows that the ARLs yielded by analytical explicit formulas were in dramatically agreement with the approximation of ARL by using numerical integral equation (NIE) method (it is different at the decimal of number 7 and so on) for all cases of this study.

The comparison of the performance of the modified EWMA and EWMA control charts were also presented. These results of ARL were evaluated by the explicit formulas as shown in Equation (8) and varying the constant $k = 0.5$, 1, 1.5, 2 of the modified EWMA control chart for exponential smoothing parameter $\lambda = 0.1$ and 0.2, as contained in Tables 2–4.

Table 2: Comparison efficiency of modified EWMA and EWMA charts by ARL for trend ARX(1,1)

δ	Modified				EWMA
	$k = 0.5$ (b=8.6554 ×10^{-3})	$k = 1$ (b=1.67697 ×10^{-2})	$k = 1.5$ (b=2.49757 ×10^{-2})	$k = 2$ (b=3.32132 ×10^{-2})	(h=1.0631 ×10^{-3})
0.00	370.0691	370.1180	370.1720	370.2023	370.1585
0.001	225.3193	184.4809	167.4260	158.2270	352.0042
0.003	126.2291	92.1460	80.0426	73.9476	320.0796
0.005	87.5518	61.4407	52.6756	48.3560	292.9218
0.01	49.4135	33.5623	28.5074	26.0623	240.0306
0.03	17.7514	12.0228	10.2581	9.4142	131.6109
0.05	10.6995	7.4010	6.3867	5.9017	85.0015
0.10	5.3117	3.8972	3.4578	3.2466	38.6232
0.30	1.9208	1.6659	1.5825	1.5415	6.6403
0.50	1.3866	1.2960	1.2651	1.2496	2.7066

Table 2 shows the ARLs of modified EWMA and EWMA control chart for the observation were from $a = 3$, $a_1 = 0.6$, $\theta = 0.5$, $b_1 = 1$, $\lambda = 0.2$ when $k = 0.5$, 1, 1.5, 2 and $ARL_0 = 370$.

Table 3 reports the ARLs of modified EWMA and EWMA control chart for the observation were from $a = 3$, $a_1 = 0.3$, $a_2 = 0.5$, $\theta = 0.5$, 1, $b_1 = 1$, $\lambda = 0.1$ when $k = 0.5$, 1, 1.5, 2 and $ARL_0 = 370$.

Table 4 shows the ARLs of modified EWMA and EWMA control chart for the observation were from $a = 3$, $a_1 = 0.4$, $a_2 = 0.6$, $\theta = 0.5$, $b_1 = 1$, $b_2 = 1$, $\lambda = 0.2$ when $k = 0.5$, 1, 1.5, 2 and $ARL_0 = 370$.

Table 3: Comparison efficiency of modified EWMA and EWMA charts by ARL for trend ARX(2,1)

δ	Modified				EWMA
	$k = 0.5$ (b=6.81292 ×10^{-3})	$k = 1$ (b=1.355403 ×10^{-2})	$k = 1.5$ (b=2.03113 ×10^{-2})	$k = 2$ (b=2.70745 ×10^{-2})	(h=2.1746 ×10^{-5})
0.00	370.0423	370.0584	370.1450	370.1508	370.0521
0.001	253.6150	196.1504	173.2719	161.3662	364.3502
0.003	155.2937	133.4250	84.0587	75.9869	353.2429
0.005	111.6570	68.0758	55.5481	49.7864	342.5184
0.01	65.2136	37.4651	30.1385	26.8569	317.2875
0.03	23.7965	13.3904	10.8080	9.6710	235.4527
0.05	14.2027	8.1786	6.6930	6.0393	176.8137
0.10	6.8021	4.2227	3.5798	3.2952	90.6454
0.30	2.1749	1.7180	1.5962	1.5408	11.1145
0.50	1.4730	1.3114	1.2659	1.2448	3.0123

Table 4: Comparison efficiency of modified EWMA and EWMA charts by ARL for trend ARX(2,2)

δ	Modified				EWMA
	$k = 0.5$ (b=2.13234 ×10^{-3})	$k = 1$ (b=4.13091 ×10^{-3})	$k = 1.5$ (b=6.15199 ×10^{-3})	$k = 2$ (b=8.18081 ×10^{-3})	(h=2.6199 ×10^{-4})
0.00	370.0449	370.0449	370.0191	370.0904	370.0681
0.001	204.8479	163.1407	146.3418	137.4444	348.3174
0.003	108.0092	77.0638	66.3908	61.0815	311.0617
0.005	73.2118	50.4762	43.0155	39.3719	280.3280
0.01	40.3882	27.1426	22.9811	20.9798	222.8748
0.03	14.2067	9.6235	8.2222	7.5542	114.4909
0.05	8.5135	5.9269	5.1362	4.7589	71.2072
0.10	4.2189	3.1517	2.8215	2.6632	30.6408
0.30	1.6043	1.4354	1.3802	1.3531	4.8124
0.50	1.2276	1.1737	1.1554	1.1462	2.0255

The results in Tables 2–4 show that the performance of the modified EWMA control chart was smaller than the EWMA control chart for all cases of both k constant and shift size. These indicated the proposed procedure is effectiveness more than the standard EWMA control chart.

4.2 *Applications to real observations*

Two real datasets were used in the study: the first comprised gold prices and the US exchange rate as the dependent and exogenous variables, respectively, while the other comprised silver prices as the dependent variable and gold prices and the Euro exchange rate as two exogenous variables. First and foremost, autocorrelation in the observations was assessed by using the Box-Jenkins technique to determine the fitting of forecast time series data models. Next, applying the t-statistic proved that the datasets were suitable for an ARX(p,r) model with a trend. Following this, we verified that the white noise followed an exponential distribution.

Dataset 1 comprised 23 points of gold prices and US exchange rates from 1 July to 31 July 2020. We proved that the dataset is an autocorrelated time series suitable for an ARX(1,1) process with a trend. Its statistics were the mean parameter a = 4,151.943, coefficient of the first-order autoregressive process a_1 = 0.611501, slope parameter θ = 11.40148, and parameter of regression exogenous variable b_1 = –76.76997. The residual of the model was confirmed as exponential white noise with the mean β_0 = 19.26 by applying the Kolmogorov-Smirnov statistic. ARL values were evaluated under various combinations of k = 0.5, 1, 1.5, 2; and exponential smoothing parameter λ = 0.1.

The results reported in Table 5 show that the ARL values on the modified EWMA control chart were smaller than those on the standard one for all k, meaning that the modified EWMA chart performed better when detecting variations in the gold price.

For dataset 2, data were recorded daily from 17 August 2020 to 30 September 2020. Analysis of these data showed that they were autocorrelated and were suitable for an ARX(1,2) model with a trend. The parameter values of the process were a_1 = 0.754476 and θ = –0.089571, while those for the independent exogenous variables were b_1 = 0.027092 and b_2 = –0.652291. Once again, we proved that the white noise was exponentially distributed by applying the Kolmogorov-Smirnov statistic to the residual term of the model (the mean β_0 = 0.2623).

Table 6 contains a comparison of the ARLs on the modified and traditional EWMA control charts derived by using explicit formulas for constant k = 0.5, 1, 1.5, 2, and 0.15. The results show that the modified EWMA control chart could detect small shifts in the process mean more sensitively because it provided smaller ARL values than the standard EWMA control chart, thereby verifying that the modified EWMA control chart is excellent for monitoring uncommon observations with unwanted values in due time for all cases of k.

Table 5: Comparison efficiency of modified EWMA and EWMA charts by ARL for trend ARX(1,1)

δ	Modified				EWMA
	k = 0.5 (b=5.51658 $\times10^{-7}$)	k = 1 (b=1.01536 $\times10^{-6}$)	k = 1.5 (b=1.479066 $\times10^{-6}$)	k = 2 (b=1.942775 $\times10^{-6}$)	(h=8.80383 $\times10^{-8}$)
0.00	370.0507	370.1869	370.0633	370.0910	370.0449
0.001	275.7573	275.8477	275.7836	275.8032	278.8191
0.003	182.7606	182.8133	182.7907	182.8015	186.8315
0.005	136.7408	136.7775	136.7679	136.7752	140.5495
0.01	84.0317	84.0525	84.0514	84.0556	86.9054
0.03	33.3163	33.3239	33.3253	33.3268	34.6392
0.05	20.9368	20.9414	20.9426	20.9435	21.7828
0.10	11.0404	11.0428	11.0435	11.0439	11.4773
0.30	4.1374	4.1382	4.1385	4.1387	4.2744
0.50	2.7364	2.7369	2.7371	2.7372	2.8113

Table 6: Comparison efficiency of modified EWMA and EWMA charts by ARL for trend ARX(1,2)

δ	Modified				EWMA
	k = 0.5 (b= 0.146415)	k = 1 (b= 0.392295)	k = 1.5 (b= 0.665839)	k = 2 (b= 0.9595)	(h= 0.0355746
0.00	370.4859	370.3579	370.0928	370.0925	370.0718
0.001	114.1169	138.6174	154.6998	166.4151	293.2146
0.003	48.4666	62.1756	72.0564	79.7338	206.0693
0.005	31.0832	40.4072	47.2843	52.7201	158.0054
0.01	16.7605	21.9477	25.8215	28.9136	98.3786
0.03	6.5134	8.4386	9.8455	10.9519	36.4188
0.05	4.3858	5.6004	6.4629	7.1271	21.0531
0.10	2.7622	3.4246	3.8685	4.1957	9.4167
0.30	1.6396	1.9064	2.0666	2.1750	2.8021
0.50	1.3998	1.5756	1.6770	1.7434	1.8714

Tables 5 and 6 contain the ARL values at various k values. The smallest ARL value for the modified EWMA control chart was exhibited for k = 0.5, which is thus the optimum value of k. The daily price of gold can be used as an indicator of an impending crisis due to all fall in the GDP growth of the United States [29], as it reveals the true state of the US economy. Increasing gold price signals that the economy is in dire straits. Gold is usually bought by investors as protection from either an economic crisis or inflation. A low gold price means that the economy is relatively secure and that investing in stocks, bonds, and real estate is likely to be profitable. In recent years, the demand for gold has tended to increase. After the recent financial crisis, investors have been seeking better investment security, and consequently, the price of gold has increased once again under these conditions.

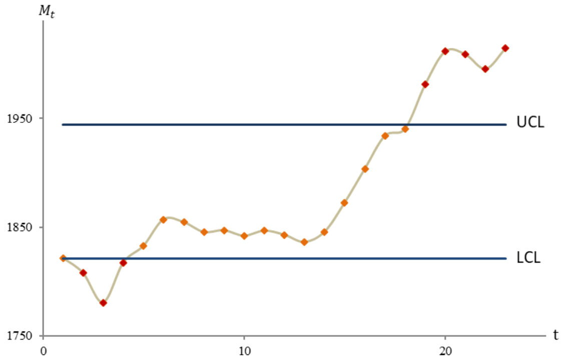

Figure 1: The detection of the modified EWMA chart for the trend ARX(1,1) model.

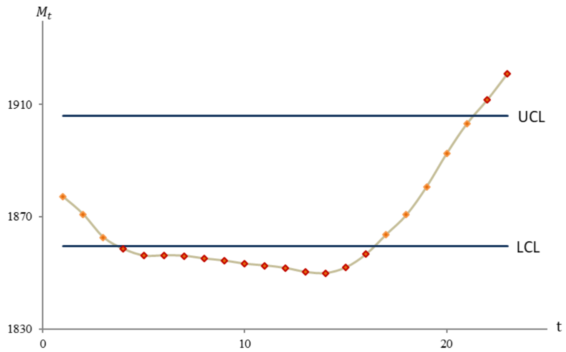

Figure 2: The detection of the standard EWMA chart for the trend ARX(1,1) model.

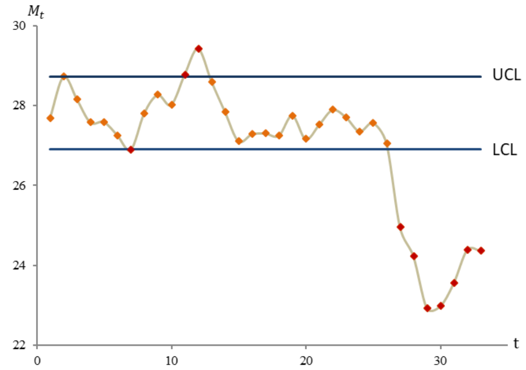

Figure 3: The detection of the modified EWMA chart for the trend ARX(1,2) model.

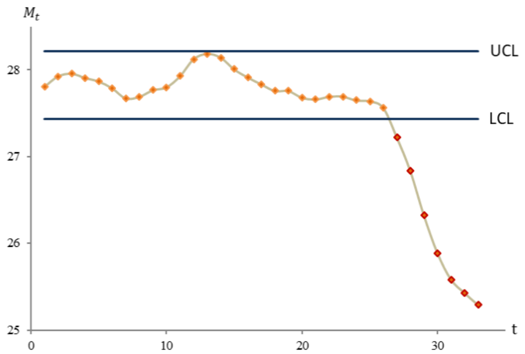

Figure 4: The detection of the standard EWMA chart for the trend ARX(1,2) model.

In the first example, there were 23 observational points of the daily US gold price from 1st July 2020 to 31st July 2020. The mean and standard deviation of this time series dataset were 1882.687 and 67.55668, respectively. The LCL and UCL of the ARL on the modified EWMA control chart were calculated using Equations (5) and (6) with the control statistic from Equation (4) and those for the standard EWMA control chart with Equations (2) and (3) with the control statistic from Equation (1), respectively. The detection of shifts in the process with the sample dataset is displayed in Figures 1 and 2.

Silver price detection using dataset 2 also illustrated the capability of the proposed procedure on the modified and standard EWMA control charts. There were 33 daily observation points for the silver price from 17th August 2020 to 30th September 2020. The mean and standard deviation of this time series dataset were 27.822 and 0.553, respectively. The LCLs and UCLs of the ARLs for both charts were established as explained previously; the performances of the ARL on the modified EWMA and traditional EWMA control charts are displayed in Figures 3 and 4, respectively.

The results in Figure 1 demonstrate that the modified EWMA control chart detected the lower gold price shifts at the 2nd to 4th observations and detected the upper gold price shifts at 19th to 23rd observations. Meanwhile, the standard EWMA control chart detected the lower gold price shifts at 4th to 16th observations and detected the upper gold price shifts at 22nd to 23rd observations, as shown in Figure 2.

The excellent performance of the modified EWMA control chart over the standard one is illustrated in Figures 3 and 4. The modified EWMA control chart detected the highest silver price shifts at the 11th and 12th observations (Figure 3). However, the lowest silver price shift was detected at the 27th to the 33rd observations by both charts (Figures 3 and 4).

5 Discussion and Conclusions

The ARL for an ARX(p,r) model with a trend and exponentially distributed white noise on the modified EWMA control chart was derived by using the explicit formulas. The accuracy of the proposed procedure, which is easy to calculate, was checked by comparing it with the ARL derived by using the NIE method. The results were in excellent agreement. The effectiveness of the ARL derived using explicit formulas on the modified EWMA control chart was demonstrated by comparison with the EWMA control chart, as determined by the smallest ARL value. The excellent performance of the ARL on the modified EWMA control chart is also illustrated by using datasets of changes in gold and silver prices comprising autocorrelated observations that fit ARX(1,1) and ARX(1,2) processes, both with a trend and exponentially distributed white noise, respectively. The empirical results indicate that the ARL derived by using explicit formulas performed better on the modified EWMA control chart than on the standard one for both processes.

Determining the optimal parameter values for the modified EWMA control chart should not be disregarded. Hence, exponential smoothing parameter values of 0.1 and 0.2 were trialed on the modified EWMA control chart; their effects on its performance were not significantly different. Meanwhile, $k = 2$ yielded the most sensitive response by the ARL on the modified EWMA control chart whereas $k = 0.5$ provided the smallest ARL value. Thus, the latter value of constant k on the modified EWMA control chart is recommended because it produced the least variance.

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References

[1] H. Schmidbauer and A. Rosch, "The impact of festivities on gold price expectation and volatility," *International Review of Financial Analysis*, vol. 58, pp. 117–131, 2018.

[2] Y. Qian, D. A. Ralescu, and B. Zhang, "The analysis of factors affecting global gold price," *Resources Policy*, vol. 64, p. 101478, 2019.

[3] D. G. Baur and T. K. J. McDermott, "Why is gold a safe haven," *Journal of Behavioral and Experimental Finance*, vol. 10, pp. 63–71, 2016.

[4] Y. Zhao, V. Upreti, and Y. Cai, "Stock returns, quantile autocorrelation, and volatility forecasting," *International Review of Financial Analysis*, vol. 73, Jan. 2021, Art. no. 101599.

[5] W.-J. Xue and L.-W. Zhang, "Stock return autocorrelations and predictability in the Chinese stock market-evidence from threshold quantile autoregressive models," *Economic Modelling*, vol. 60, pp. 391–401, Jan. 2017.

[6] M.-S. Pan, K. Liano, and G.-C. Huang, "Industry momentum strategies and autocorrelations in stock returns," *Journal of Empirical Finance*, vol. 11, pp. 185–202, Mar. 2004.

[7] X. Dong, S. Feng, L. Ling, and P. Song, "Dynamic autocorrelation of intraday stock returns," *Finance Research Letters*, vol. 20, pp. 274–280, Feb. 2017.

[8] M. M. Rounaghi and F. N. Zadeh, "Investigation of market efficiency and financial stability between S&P 500 and London stock exchange: Monthly and yearly forecasting of time series stock returns using ARMA model," *Physica A: Statistical Mechanics and its Application*, vol. 456, pp. 10–21, Aug. 2016.

[9] J. T. Linnainmaa, W. Torous, and J. Yae, "Reading the tea leaves: Model uncertainly, robust forecasts, and the autocorrelation of analysts' forecast errors," *Journal of Financial Economics*, vol. 122, pp. 42–64, Jan. 2016.

[10] E. S. Page, "Continuous inspection schemes," *Biometrika*, vol. 41, no. 1–2, pp. 100–115, Jun. 1954.

[11] S. W. Robert, "Control chart test based on geometric moving average," *Technometrics*, vol. 1, pp. 239–250,1959.

[12] A. K. Patel and J. Divecha, "Modified exponentially weighted moving average (EWMA) control chart for an analytical process data," *Jornal of Chemical Engineering and Materials Science*, vol. 2, pp. 12–20, 2011.

[13] N. Khan, M. Aslam, and C. Jun, "Design of a control chart using a modified EWMA statistic,"

Quality and Reliability Engineering International, vol. 41, pp. 100–114, 2017.

[14] M. Aslam, A. Saghir, L Ahmad, and C. H. Hussain, "A control chart for COM-Poisson distribution using a modified EWMA statistic," *Journal of Statistical Computation and Simulation*, vol. 87, no. 18, pp. 3491–3502, 2017.

[15] E. T. Herdiani, G. Fandrilla, and N. Sunusi, "Modified exponential weighted moving aveage (EWMA) control chart on autocorrelation data," *Journal of Physics Conference Series*, vol. 979, 2018, Art. no. 012097.

[16] T. Noiplab and T. Mayureesawan, "Modified EWMA control chart for skewed distribution and contaminated process," *Thailand Statistician*, vol. 17, pp. 16–29, 2019.

[17] A. Saghir, A. Ahman, and M. Aslam, "Modified EWMA control chart for transformed gamma data," *Communications in Statistics-Simulation and Computation*, May 2019, doi: 10.1080/03610 918.2019.1619762.

[18] M. Aslam and S. M. Anwar, "An improved Bayesian Modified EWMA location chart and it applications in mechanical and sport industry" *PLOS ONE*, vol. 15, 2020, Art. no. e0229422.

[19] Y. Supharakonsakun, Y. Areepong, and S. Sukparungsee, "The exact solution of the average run length on a modified EWMA control chart for the first-order moving-average process," *ScienceAsia*, vol. 46, pp. 109–118, 2020.

[20] Y. Supharakonsakun, Y. Areepong, and S. Sukparungsee, "The performance of a modified EWMA control chart for monitoring autocorrelated PM2.5 and carbon monoxide air pollution data," *PeerJ*, vol. 8, 2020, Art. no. e10467.

[21] P. Paichit, "Exact expression for average run length of control chart of ARX(p) procedure," *KKU Science Journal*, vol. 45, no. 4, pp. 948–958, 2017.

[22] Y. Areepong, "The exact solution of ARL on EWMA chart for nonseasonal AR model with exogenous," *Advances and Applications in Statistics*, vol. 57, pp. 41–59, Jul. 2019.

[23] W. Peerajit and Y. Areepong, "The performance of cusum control chart for monitoring process mean for autoregressive moving average with exogenous variable model," *Applied Science and Engineering Progress*, 2020, Art. no. 3490, doi: 10.14416/j.asep.2020.11.007.

[24] R. Sunthornwat and Y. Areepong, "Average run length on CUSUM control chart for seasonal and non-seasonal moving average process with exogenous variables," *Symmetry*, vol. 12, no. 1, p. 173, Jan. 2020.

[25] P. Phanthuna, Y. Areepong, and S. Sukparungsee, "Detection capability of the modified EWMA chart for the trend stationary AR(1) model," *Thailand Statistician*, vol. 19, no. 1, pp. 70–81, Jan. 2021.

[26] D. C. Montgomery, *Introduction to Statistical Quality Control*. 6th ed., Hoboken, NJ: Wiley. Chicago, 2009, pp. 423.

[27] C. W. Champ and S. E. Rigdon, "A comparison of the markov chain and the integral equation approaches for evaluating the run length distribution of quality control charts," *Communications in Statistics-Simulation and Computation*, vol. 21, no. 1, pp. 191–204, 1991.

[28] Wolfram Research, "WOLFRAM MATHE MATICA 8.0.," 1998. [Online]. Available: https://www.wolfram.com/mathematica/

[29] M. Bildirici, F. Kay1kc1, and I. S. Onat, "BDI, Gold price and economic growth," *Procedia Economics and Finance*, vol. 38, pp. 280–286, 2016.